Product of n independent Uniform Random Variables

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We give an alternative proof of a useful formula for calculating the probability density function of the product of n uniform, independently and identically distributed random variables. Ishihara (2002, in Japanese) proves the result by induction; here we use Fourier analysis and contour integral methods which provide a more intuitive explanation of how the convolution theorem acts in this case.

To obtain the probability density function (PDF) of the product of two continuous random variables (r.v.) one can take the convolution of their logarithms. This is explained for example by Rohatgi (1976). It is possible to use this repeatedly to obtain the PDF of a product of multiple but fixed number ($n > 2$) of random variables. This is however a very lengthy process, even when dealing with uniform distributions supported on the interval $[a, b]$. We encountered the latter problem with $a = \frac{1}{3}$ and $b = 3$, in the article by Armstead *et al.* (2004) on the approximation for the open-ended stadium billiard dynamical system; there are undoubtedly other applications in a variety of fields. A formula for calculating the PDF of the product of n uniform independently and identically distributed random variables on the interval $[0, 1]$ first appeared in Springer's book (1979) on "The algebra of random variables". This was then generalized (see Ishihara 2002 (in Japanese)) to accommodate for independent but not identically (i.e. $\{[a_i, b_i], i = 1, 2, \dots n\}$) distributed uniform random variables through the use of the proof by induction. In the current paper we use Fourier analysis, as suggested by Springer, to re-derive a subset of Ishihara's results: the PDF of a product of n independent and identically distributed uniform [a, b] random variables. Through this analysis one can see exactly how the n smooth components of the resulting PDF arise from contour integrals in Fourier space and thus obtain a more intuitive idea of how the convolution theorem (see Bracewell, 2000) acts. Specifically, we shall show that the convergence of the contour integrals defines the supports of the components of the PDF.

Theorem 1. *Let X_i be independent random variables with PDF $f_{X_i}(x) = \frac{1}{b-a}$ on the interval $x \in [a, b]$ and 0 otherwise, where $0 \leq a < b < \infty$ and $i = 1, 2, \dots n$, $n \geq 2$. Then the PDF of $X = \prod_{i=1}^{n} X_i$ is given by the piecewise smooth function:*

$$f_X(x) = \begin{cases} f_X^k(x), & a^{n-k+1}b^{k-1} \leq x \leq a^{n-k}b^k, \\ & k = 1, 2, \dots n, \\ 0, & \text{otherwise,} \end{cases}$$

where

$$f_X^k(x) = \sum_{j=0}^{n-k} \frac{(-1)^j}{(b-a)^n (n-1)!} \binom{n}{j} \left(\ln \frac{b^{n-j}a^j}{x} \right)^{n-1}.$$

Remark 1. It is interesting to note that the components' derivatives $(\frac{\mathrm{d}^l}{\mathrm{d}x^l} f_X^k(x))$, of order $l = 1, 2, \dots (n-2)$, are continuous at their end-points while the $(n-1)_{th}$ derivative is not (see Springer 1979).

Remark 2. The known result that $\ln X = \ln \prod_{i=1}^{n} X_i = \sum_{i=1}^{n} \ln X_i$ is Gamma distributed $(\sim -\Gamma(n, 1))$, as explained by Devroye, (1986), is only valid for $a = 0$, with the natural normalization $b = 1$. Unfortunately, we can not find a representation in terms of standard distributions if $a > 0$. We can however comment that according to the Central Limit Theorem (CLT), the distribution of $\ln X$ converges asymptotically to the Normal distribution. In fact, since the third central moment of $\ln X_i$ exists and is finite, then by the Berry-Essen theorem (see Feller 1972), the convergence is uniform and the the convergence rate is at least of the order of $1/\sqrt{n}$; this can be used to approximate $f_X(x)$ for large n where direct numerical computation is inefficient.

Proof. Let $Y_i = \ln X_i$. Then the PDF of Y_i is $f_{Y_i}(y) = \frac{1}{b-a} e^y = \kappa e^y$ supported on $y \in (\ln a, \ln b)$ and is zero otherwise.

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We find the characteristic function by taking the Fourier transform of $f_{Y_i}(y)$:

$$\begin{aligned}\mathcal{F}(f_{Y_i}(y))(\eta) = \mathbb{E}(e^{i\eta Y_i}) = \hat{f}_{Y_i}(\eta) &= \int_{-\infty}^{\infty} \kappa e^{y} e^{i\eta y} \mathrm{d}y, \\ &= \frac{\kappa}{(1+i\eta)}\Big(be^{i\eta \ln b} - ae^{i\eta \ln a}\Big). \end{aligned} \tag{1}$$

The convolution theorem (see Bracewell, 2000) states that the characteristic function (c.f.) of the sum of n random variables is given by the product of the individual c.f. of each r.v. Hence, the c.f. of $Y = \sum_{i=1}^{n} Y_i$ is given by the nth power of $\hat{f}_{Y_i}(\eta)$ which we expand here using the binomial theorem:

$$[\hat{f}_{Y_i}(\eta)]^n = \hat{f}_Y(\eta) = \sum_{j=0}^{n} \frac{\kappa^n(-1)^j}{(1+i\eta)^n}\binom{n}{j} b^{(n-j)} a^j e^{i\eta\lambda_j} \tag{2}$$

where $\lambda_j = (n-j)\ln b + j\ln a$. To perform the inverse Fourier transform we shall use Cauchy's residue theorem (see Knopp, 1996). Note that according to Springer (1979), we should expect n piecewise continuous components which make up a C^{n-2} curve. Also note that the inverse Fourier transform of equation (2), $\mathcal{F}^{-1}\big([\hat{f}_{Y_i}(\eta)]^n\big)(y)$, will have support only in the interval $(n\ln a, n\ln b)$.

$$\begin{aligned}\mathcal{F}^{-1}\big([\hat{f}_{Y_i}(\eta)]^n\big)(y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}_Y(\eta) e^{-i\eta y}\ \mathrm{d}\eta \\ &= \int_{-\infty}^{\infty} \sum_{j=0}^{n} \frac{\kappa^n(-1)^j\binom{n}{j} b^{(n-j)} a^j e^{i\eta(\lambda_j - y)}}{2\pi(\eta - i)^n (i)^n}\ \mathrm{d}\eta \\ &\equiv \int_{-\infty}^{\infty} \sum_{j=0}^{n} h_j(\eta, y)\ \mathrm{d}\eta, \end{aligned} \tag{3}$$

where the integral-sum order can be interchanged. We define two contours γ_m ($m = 1, 2.$) such that γ_1 goes along the real axis from $-R$ to R and then into the upper complex plane along an anti-clockwise semicircular arc of radius $R > 1$, centered at the origin, $\gamma_{c_1} \subset \gamma_1$. Contour γ_2 is defined similarly but into the lower complex plane along a clockwise semicircular arc of radius R, $\gamma_{c_2} \subset \gamma_2$. Notice that for all j there is only one pole due to $h_j(\eta, y)$ enclosed by γ_1, that it is of order n, that it is situated at $\eta_0 = i$ and that there are no poles in γ_2. We use the residue theorem to calculate:

$$\begin{aligned}\oint_{\gamma_1} h_j(\eta, y)\ \mathrm{d}\eta &= 2\pi i \mathrm{Res}\Big(h_j(\eta, y), i\Big) \\ &= \frac{(\kappa)^n(-1)^j}{(n-1)!}\binom{n}{j}\big(\lambda_j - y\big)^{(n-1)} e^{y}. \end{aligned} \tag{4}$$

The choice of contour to be used for every $0 \le j \le n$ and $y \in (n\ln a, n\ln b)$ when calculating (3) depends on the sign of the exponential. In other words, m depends on both j and y. Explicitly, we write $\eta = R(\cos\phi + i\sin\phi)$ and estimate the integrals over the semicircular arcs γ_{c_1} and γ_{c_2}:

$$\int_{\gamma_{c_m}} h_j(\eta, y)\mathrm{d}\eta = \int_{\gamma_{c_m}} g(R, \phi) e^{-R\sin\phi(\lambda_j - y)}\mathrm{d}\phi, \tag{5}$$

where $g(R, \phi) = \mathcal{O}(R^{-n+1})$, as $R \to \infty$. For $n \ge 2$ we know that if the exponent: $-R\sin\phi(\lambda_j - y) \le 0$, then the integrals in (5) will converge to zero. We rearrange this inequality to find that for γ_1 we need $j \le j^*(y)$ while for γ_2 we need $j > j^*(y)$, where $j^*(y) = \lfloor \frac{n\ln b - y}{\ln b - \ln a} \rfloor$ and $\lfloor . \rfloor$ denotes the floor function. Note that when $\lambda_j = y$, both contour integrals (along γ_{c_1} and γ_{c_2}) converge and we see that (4) is identically zero. Hence we obtain the following equation:

$$\begin{aligned} f_Y(y) &= \sum_{j=0}^{n} \int_{-\infty}^{\infty} h_j(\eta, y)\ \mathrm{d}\eta \\ &= \sum_{j=0}^{j^*(y)} \Big(\oint_{\gamma_1} h_j(\eta, y)\ \mathrm{d}\eta - \int_{\gamma_{c_1}} h_j(\eta, y)\ \mathrm{d}\eta\Big) \\ &\quad + \sum_{j=j^*(y)+1}^{n} \Big(\oint_{\gamma_2} h_j(\eta, y)\ \mathrm{d}\eta - \int_{\gamma_{c_2}} h_j(\eta, y)\ \mathrm{d}\eta\Big) \end{aligned} \tag{6}$$

as $R \to \infty$, where all integrals along γ_{c_1}, γ_2 and γ_{c_2} vanish and the remaining integral is given by (4). Note that the sums in (6) only make sense if $0 \leq j^*(y) < n$; as expected from the known support of y. We find n intervals on which $f_Y(y)$ is supported and number them by $k = 1, 2, \ldots n$, where $k = n - j^*(y)$. To obtain $f_X(x)$, as given in Theorem 1., simply transform back to $X = \exp(Y)$. $\square$

Remark 3. It is an interesting exercise to show that $\sum_{j=0}^{n} \left(\oint_{\gamma_m} h_j(\eta, y)\ \mathrm{d}\eta \right) = 0$ for both $m = 1$ and $m = 2$ and for any y as $R \to \infty$. To see this for $m = 1$, expand $(\lambda_j - y)^{(n-1)}$ using the binomial theorem, collect the j-dependent terms and interchange the sums to obtain:

$$
\begin{aligned}
&\sum_{j=0}^{n} \Big(\oint_{\gamma_1} h_j(\eta, y)\ \mathrm{d}\eta \Big) = \\
&\quad \sum_{l=0}^{n-1} \frac{(\kappa)^n \big(\ln \frac{a}{b}\big)^l e^y}{(n-1)!(i)^{n-1}} \binom{n-1}{l} \big(n \ln b - y\big)^{n-1-l} \\
&\quad \times \sum_{j=0}^{n} (-1)^j \binom{n}{j} j^l.
\end{aligned}
$$

To show that the last sum over j is zero, we write it as:

$$
\begin{aligned}
\sum_{j=0}^{n} (-1)^j \binom{n}{j} j^l e^{ls} \bigg|_{s=0} &= \frac{\mathrm{d}^l}{\mathrm{d}s^l} \sum_{j=0}^{n} (-1)^j \binom{n}{j} e^{ls} \bigg|_{s=0} \\
&= \frac{\mathrm{d}^l}{\mathrm{d}s^l} \big(1 - e^s\big)^n \bigg|_{s=0} = 0,
\end{aligned}
$$

for all $0 \leq l \leq (n-1)$. For $m = 2$, the contour integral is zero as there are no poles enclosed by the contour.

Remark 4. To prove Ishihara's general result (where the X_i's are not identically distributed), one would have to expand the product $\prod_{j=1}^{n} \frac{(b_j e^{i\eta \ln b_j} - a_j e^{i\eta \ln a_j})}{(b_j - a_j)}$ and evaluate the $(n-1)_{th}$ derivative at $\eta = i$, and then look at the various contour integrals as above. While possible in principle, this would defeat the purpose of this paper, namely a simpler but more explicit and intuitive derivation of the result.

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